INCOMPRESSIBLE LIMIT FOR A TWO-SPECIES MODEL WITH COUPLING THROUGH BRINKMAN'S LAW IN ANY DIMENSION

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ABSTRACT. We study the incompressible limit for a two-species model with applications to tissue growth in the case of coupling through the so-called Brinkman's law in any space dimensions. The coupling through this elliptic equation accounts for viscosity effects among the individual species. In a recent paper DĘBIEC & SCHMIDTCHEN established said result in one spacial dimension, with their proof hinging on being able to establish uniform BV-bounds. This approach is fundamentally different from the one-species case in arbitrary dimension, established by PERTHAME & VAUCHELET. Their result relies on a kinetic reformulation to obtain strong compactness of the pressure. In this paper we fill this gap in the literature and present the incompressible limit for the system in arbitrary space dimension. The difficulty stems from jump discontinuities in the pressure not only at the boundary of the support of the two species but also at internal layers giving rise to the question as to how compactness can be obtained. The answer is a combination of techniques consisting of the application of the compactness method of BRESCH & JABIN, an adaptation of the aforementioned kinetic reformulation, and several parallels to the one dimensional strategy. The main result of this paper establishes a rigorous bridge between the population dynamics of growing tissue at a density level and a geometric model thereof.

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1 INTRODUCTION

In recent years there has been an increasing interest in multi-phase models applied to tumour growth. Traditionally, tumour growth was modelled using a single equation describing the evolution of the density of tumorous cells. This paper is dedicated to studying the two-species model

$$\begin{cases} \dfrac{\partial n_k^{(i)}}{\partial t} - \nabla \cdot \left(n_k^{(i)} \nabla W_k \right) &= n_k^{(i)} G^{(i)}(p_k), \\ -\nu \Delta W_k + W_k &= p_k, \end{cases}$$

where $n^{(i)}$ represents the density of the healthy (resp. cancerous) cells, for $i = 1, 2$, and p_k, denotes the joint population pressure generated by the two species, *i.e.*,

$$p_k := \frac{k}{k-1} n_k^{k-1},$$

where

$$n_k := n_k^{(1)} + n_k^{(2)},$$

denotes the total population. The model parameters $k \in \mathbb{N}$ and $\nu > 0$ model the stiffness of the pressure and the level of viscosity, respectively. Note that the velocity field, W_k, is generated by the joint population pressure through so-called *Brinkman's law*, for instance *cf.* [1]. Unlike *Darcy's law*, the *Brinkman flow* incorporates viscosity effects into the model. In addition to the

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advection, the two cell densities are assumed to proliferate. This effect is encoded in the two functions $G^{(i)}$, for $i = 1, 2$, which are assumed to be decreasing in their variable, p_k, similar to [10, 26]. This accounts for an inhibited growth whenever the joint pressure gets too large.
The total population plays an important role in the analysis of these types of models that was exploited in many related results, *cf.* [11, 14].
It is easily verified that the total population density, n_k, satisfies

$$\frac{\partial n_k}{\partial t} - \nabla \cdot (n_k \nabla W_k) = n_k \Big(r_k G^{(1)}(p_k) + (1 - r_k) G^{(2)}(p_k) \Big), \tag{1}$$

where r_k denotes the concentration

$$r_k := n_k^{(1)} / n_k, \tag{2}$$

also referred to as the *population fraction*. Related models have been extensively studied in the past. We refer to [23, 25], and references therein, for a study of the stiff limit for a single-species visco-elastic tumour model. As above, the velocity field is given by Brinkman's law. Another approach was taken in [21] where the authors construct viscosity solutions to the same visco-elastic single-species tissue growth model. Their result extends the result of [25] in that they obtain pointwise convergence and, what is more, uniform convergence away from the boundary of the support. Similarly to [25], a part of the paper is dedicated to studying the evolution of the support as a geometric flow. However, they observe that a regularisation of the velocity field is sufficient to apply the method of characteristics, to pass to the limit later, and to obtain the same complementarity relation of the limiting pressure.
In their case, the pressure is directly linked to the density itself, rather than the sum of the two densities.
As a matter of fact, coupling the two equations for the individual species through Brinkman's law, this time generated by the *joint* population pressure, changes the behaviour dramatically and the strategy of a kinetic reformulation used in [25] cannot be applied directly. Two of the authors were able to establish the incompressible limit in the one dimensional case, *cf.* [14]. Their proof mainly relies on establishing uniform BV-bounds for the two species as well as the total population. In conjunction with the compactness criterion [18, Lemma A] they infer the strong compactness of the pressure which suffices to pass to the limit. In their paper, the authors already note that the BV-strategy fails in higher dimensions and they posit new techniques be indispensable in the quest of extending the result to higher dimensions.
Even in the inviscid case, *i.e.*, $\nu = 0$, the system nature of the problem causes serious analytical difficulties, *cf.* [9, 11, 16]. In this case, the pressure does gain a bit in regularity. Yet, this gain in regularity is just about sufficient to obtain strong compactness of the pressure gradient. We highlight that similar difficulties arise when the pressure is not given in form of a power law but blows up at a finite threshold, *cf.* [12, 13, 17]. All these results have one thing in common – their minute study of the equation satisfied by the population pressure, *cf.* [9, 11–13, 16, 17, 20–22, 24, 25], which allows for proving the existence of solutions and obtaining estimates that are uniform in the stiffness parameter, k.
The innovation of this work is to combine techniques for the one-species case in any space dimension and the two-species case in one dimension. In conjunction with a nonlocal compactness criterion, originally introduced and devised by Bresch & Jabin, *cf.* [19], the required compactness is obtained and the passage to the incompressible limit is accomplished.
This technique, just like the aforementioned results, rely on a clever choice of auxiliary variables: it is readily checked that the joint population pressure satisfies

$$\frac{\partial p_k}{\partial t} - \nabla p_k \cdot \nabla W_k = \frac{k-1}{\nu} p_k \Big[W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu (1 - r_k) G^{(2)}(p_k) \Big],$$

where the population fraction r_k, satisfies

$$\frac{\partial r_k}{\partial t} - \nabla r_k \cdot \nabla W_k = r_k (1 - r_k) \Big[G^{(1)}(p_k) - G^{(2)}(p_k) \Big].$$

This change of variables was first introduced in [3–5] in the context of a two-species system where the two species avoid overcrowding, paving the way for more modern approaches to tumour models linked through Darcy's law, *cf.* [6, 9, 11, 16].

The rest of this paper is organised as follows. In the subsequent section, Section 2, we give a precise statement of the problem. In addition, we introduce the key assumptions on the growth terms and the initial data. Finally, we state the main result of this work, the incompressible limit and the complementarity relation. Section 3 is dedicated to establishing certain a priori estimates on the individual species, the pressure, and the velocity field given through Brinkman's law. Section 4 is devoted to establishing the strong compactness of the two individual species as well as the total population. Note that this is where we deviate from the strategy of [25] and, instead, follow [28]. A key step in the argument is to study not only the individual species but to include the total population in the estimates. At first glance, this approach may seem rather absurd and the reader might wonder what could possibly be gained by incorporating the total population into the estimates that has not already been obtained from the two individual species. However, this strategy already proved most useful in [14] as certain cancellations can be obtained in doing so. Having established the strong compactness of the individual species, we address the compactness of the pressure in Section 5. Here, we borrow the technique of the kinetic reformulation from [25] to lift the strong compactness of the individual species to the joint population pressure. In Section 6 we gather all preceding results and establish the proof of the main result of this work, *i.e.*, the incompressible limit and the complementarity relation.

2 Preliminaries and Statement of the Main Results

We study the system

$$\begin{cases} \dfrac{\partial n_k^{(1)}}{\partial t} - \nabla \cdot \left(n_k^{(1)} \nabla W_k\right) = n_k^{(1)} G^{(1)}(p_k), \\ \dfrac{\partial n_k^{(2)}}{\partial t} - \nabla \cdot \left(n_k^{(2)} \nabla W_k\right) = n_k^{(2)} G^{(2)}(p_k), \end{cases} \tag{3a}$$

posed on $(0,T)\times\mathbb{R}^d$. It is coupled through an elliptic equation, the so-called Brinkman's law

$$-\nu \Delta W_k + W_k = p_k. \tag{3b}$$

The system is equipped with non-negative initial data

$$n_{0,k}^{(i)} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \tag{4a}$$

for any $k \geq 2$, such that

$$p_{0,k} \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d), \quad \text{as well as} \quad p_{0,k} \leq P_H,$$

for some positive constant $P_H > 0$. In addition, we assume that the initial data are compact, *i.e.*, for $i=1,2$, there exist non-negative functions $n_{0,\infty}^{(i)} \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, such that

$$n_{0,k}^{(i)} \longrightarrow n_{0,\infty}^{(i)}, \tag{4b}$$

strongly in $L^1((0,T)\times\mathbb{R}^d)$, as $k\to\infty$.
As before, the pressure is given in form of a power of the joint population, *i.e.*,

$$p_k := \frac{k}{k-1}\left(n_k^{(1)} + n_k^{(2)}\right)^{k-1} = \frac{k}{k-1} n_k^{k-1}. \tag{5}$$

Recall that the pressure satisfies

$$\frac{\partial p_k}{\partial t} - \nabla p_k \cdot \nabla W_k = \frac{k-1}{\nu} p_k \Big[W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1-r_k) G^{(2)}(p_k)\Big], \tag{6}$$

with the population fraction, $r_k := n_k^{(1)}/n_k$, given by

$$\frac{\partial r_k}{\partial t} - \nabla r_k \cdot \nabla W_k = r_k(1-r_k)\Big[G^{(1)}(p_k) - G^{(2)}(p_k)\Big]. \tag{7}$$

Throughout the paper we assume the following regularity and properties of the growth functions, $G^{(i)}$,

$$G^{(i)} \in C^1(\mathbb{R}), \quad G_p^{(i)} \leq -\alpha < 0, \quad \text{and} \quad \max_{i=1,2} G^{(i)}(P_H) = 0, \tag{8}$$

for $i = 1, 2$, and some $\alpha > 0$, where $G_p^{(i)}$ denotes the derivative of the function $G^{(i)}$ with respect to the pressure. We remark that the constant P_H is usually referred to as the *homeostatic pressure*.

Remark 2.1 (Existence of solutions)**.** Let us point out that existence of bounded weak solutions to System (3) has been recently established by DĘBIEC & SCHMIDTCHEN, *cf.* [14], using a fixed point argument. For the reader's convenience we recall here the precise statement.

Theorem 2.2. *Let $T > 0$. For any initial data satisfying (4), System (3) admits a unique distributional solution $(n_k^{(1)}, n_k^{(2)})$, with $n_k^{(i)} \in L^\infty(0, T; L^\infty(\mathbb{R}^d))$.*

In fact, in this paper we establish compactness of the sequences of individual densities, which would allow for another proof of existence, via a viscosity approximation approach, see also [2].

Below we formulate the main result of the current work.

Theorem 2.3 (Incompressible limit and complementarity relation)**.** *We may pass to the limit $k \to \infty$ in the pressure equation, Eq. (6), using strong compactness of the pressure and the two species. This yields the so-called complementarity relation*

$$p_\infty \Big[W_\infty - p_\infty + \nu n_\infty^{(1)} G^{(1)}(p_\infty) + \nu n_\infty^{(2)} G^{(2)}(p_\infty) \Big] = 0, \tag{9}$$

almost everywhere, where $n_\infty^{(i)}$, satisfies

$$\begin{cases} \dfrac{\partial n_\infty^{(i)}}{\partial t} - \nabla \cdot \left(n_\infty^{(i)} \nabla W_\infty \right) &= n_\infty^{(i)} G^{(i)}(p_\infty), \\ -\nu \Delta W_\infty + W_\infty &= p_\infty, \end{cases}$$

in the distributional sense, for $i = 1, 2$. Moreover, the following holds true almost everywhere

$$p_\infty(n_\infty - 1) = 0.$$

In other words, Theorem 2.3 provides a rigorous link between the description of the evolution of the two populations, *cf.* Eq. (3a), and a geometric free-boundary model of Hele-Shaw flavour. Regions with positive pressure correspond to fully saturated areas, since $p_\infty(n_\infty - 1) = 0$, and on such domains, the pressure is given by the so-called *complementarity relation*, *cf.* Eq. (9). The rigorous derivation is already known in one spatial dimension, *cf.* [14] and, for one species, in any dimension, *cf.* [25]. In both works the authors emphasised the possibility of jump discontinuities in the pressure which renders the problem of obtaining compactness rather challenging. An extension of the strategy of [14] to higher dimensions appears futile, as does the extension of [25] to two species due to the contribution of the individual species and their role in the identification of weak-$\star$ limits in the kinetic reformulation. The subsequent sections are concerned with the proof of the main theorem.

3 A Priori Estimates

The proof of Theorem 2.3 relies on certain uniform bounds for the main quantities of interests, *i.e.*, the population densities, the pressure, and the velocity field. These will be vital when passing to the limit $k \to \infty$. The proofs of most of these assertions are rather straightforward, and have been carefully written in previous works of the authors, *cf.* [14, 25]. We therefore skip them here for brevity.

The first lemma establishes uniform bounds for the densities and the pressure.

Lemma 3.1 (A priori estimates I)**.** *The following hold uniformly in k for any $T > 0$.*

(i) $n_k^{(i)} \geq 0$, *for* $i = 1, 2$,

(ii) $n_k \in L^\infty(0,T;L^1(\mathbb{R}^d))$,

(iii) $p_k \in L^\infty(0,T;L^\infty(\mathbb{R}^d))$, *with* $0 \le p_k \le P_H$,

(iv) $n_k \in L^\infty(0,T;L^\infty(\mathbb{R}^d))$,

(v) $p_k \in L^\infty(0,T;L^1(\mathbb{R}^d))$, *and*

(vi) $n_k^{(i)} \in L^\infty(0,T;L^\infty(\mathbb{R}^d))$, *for* $i=1,2$.

The second lemma states some useful observations regarding regularity of the sequence $(W_k)_k$. Recall that a solution W_k to Brinkman's equation, Eq. (3b), may be written as $W_k = \mathfrak{K}\star p_k$, where $\mathfrak{K}$ is the fundamental solution to the equation $-\nu\Delta\mathfrak{K} + \mathfrak{K} = \delta_0$, *i.e.*,

$$\mathfrak{K}(x) = \frac{1}{4\pi}\int_0^\infty \exp\left[-(\pi|x|^2/4s\nu + s/4\pi)\right] s^{-d/2}\,\mathrm{d}s.$$

Then $\mathfrak{K} \ge 0$, $\int \mathfrak{K}(x)\,\mathrm{d}x = 1$ and

$$\mathfrak{K} \in L^q(\mathbb{R}^d),$$

for $1 \le q < d/(d-2)$, as well as,

$$\nabla\mathfrak{K} \in L^q(\mathbb{R}^d),$$

for $1 \le q < d/(d-1)$. By the elliptic regularity theory we have $W_k \in L^\infty\big(0,T;W^{2,q}(\mathbb{R}^d)\big)$, for $1 \le q \le \infty$.

Lemma 3.2 (A priori estimates II). *The following hold uniformly in k for any $T>0$.*

(i) $W_k \in L^\infty(0,T;L^1\cap L^\infty(\mathbb{R}^d))$,

(ii) $W_k \in L^\infty(0,T;W^{1,q}(\mathbb{R}^d))$, *for* $1 \le q \le \infty$,

(iii) $\nabla^2 W_k \in L^\infty(0,T;L^q(\mathbb{R}^d))$, *for* $1 < q < \infty$,

(iv) $\partial_t W_k \in L^1(0,T;L^q(\mathbb{R}^d))$, *for* $1 \le q \le \infty$, *and*

(v) $\partial_t \nabla W_k \in L^1(0,T;L^q(\mathbb{R}^d))$, *for* $1 < q < d/(d-1)$.

Using the above lemma and the boundedness of W_k, we have the following result.

Lemma 3.3 (Integrability and Segregation). *If both species are segregated initially,* i.e.,

$$r_{0,k}(1-r_{0,k}) = 0,$$

almost everywhere, then there holds

$$r_k(t,x)\,(1-r_k(t,x)) = 0,$$

almost everywhere, for all times $0 \le t \le T$. *Moreover,* $r_{0,k}(1-r_{0,k}) \in L^1(\mathbb{R}^d)$ *implies* $r_k(1-r_k) \in L^\infty(0,T;L^1(\mathbb{R}^d))$.

It is easy to infer from the preceding lemma that the population fraction itself is integrable.

Remark 3.4 (The population fraction is locally integrable). The population fraction r_k is bounded in $L^\infty(0,T;L^1_{loc}(\mathbb{R}^d))$.

Finally, the following lemma establishes an L^1-bound on the right-hand side of the pressure equation.

Lemma 3.5 (A priori estimates III). *The following estimate holds for any $T>0$*

$$k\int_0^T\int_{\mathbb{R}^d} p_k|W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1-r_k)G^{(2)}(p_k)|\,\mathrm{d}x\,\mathrm{d}t \le C(T),$$

for a constant $C(T)$, independent of k.

This estimate follows directly from studying the dissipation of the quantity

$$\int_{\mathbb{R}^d} \left| W_k - p_k + \nu n_k^{(1)} G^{(1)}(p_k) + \nu n_k^{(2)} G^{(2)}(p_k) \right| \mathrm{d}x,$$

along solutions of System (3). Since the proof is parallel to the one dimensional case, we refer the reader to [14, Lemma 4.3] for a detailed derivation of this bound.

4 Strong Compactness of the Densities

The aim of this section is to establish local strong compactness of the two individual species, *i.e.*, $(n_k^{(i)})_k$, for $i = 1, 2$. To this end, we follow the strategy originally proposed by Bresch & Jabin, *cf.* [8], (see also [7]) in the context of compressible Navier-Stokes equations with non-monotone pressure law and anisotropic stress tensor. Their approach has also been adapted to the case of the whole space, $\mathbb{R}^d$, with a growth term on the right-hand side of the continuity equation in [28].
Here we adapt their strategy to the case of a system of two interacting species, where the interaction is given by the elliptic Brinkman's law and the individual growth terms. The main result of this section reads

Theorem 4.1 (Compactness of the species). *Suppose the initial data $n_{0,k}^{(i)}$ is compact, in the sense of Eq. (4b), and suppose the growth terms satisfy Eq. (8). Then both the individual species and the total population density $(n_k)_k$, $(n_k^{(1)})_k$, $(n_k^{(2)})_k$ are compact in $L^1_{loc}\big((0,T)\times\mathbb{R}^d\big)$.*

In particular, this will be used to conclude the following.

Corollary 4.2 (Strong compactness of r_k). *The sequence $(r_k)_k$, of population fractions, is compact in $L^p\big(0,T,L^q_{loc}(\mathbb{R}^d)\big)$, for any $1 \le p, q < \infty$.*

Remark 4.3. It is worth pointing out that even though our ultimate objective is to obtain strong compactness of the pressure, we cannot apply the method presented below directly to the pressure in order to accomplish this task. This is due to the fact that the right-hand side of the equation satisfied by the pressure, *cf.* Eq. (6), is merely in L^1, *cf.* Lemma 3.5, which turns out to be borderline for obtaining estimates independent of the parameter k. As we will see, to establish compactness of the pressure, we need compactness of the densities, $n_k^{(i)}$, and the remarkable aspect of Theorem 4.1 is to prove compactness of the densities when the pressure is merely bounded (and not necessarily compact).

For clarity of exposition, throughout this section we will omit time dependence in the quantities of interest, writing $n_k(x)$, rather than $n_k(t,x)$.

4.1 The Compactness Criterion.

We begin the proof of Proposition 4.1 by setting out the compactness criterion of Jabin et al. We omit the proof, referring the reader to [2, Lemma 3.1], or [8, Proposition 4.1], for a detailed motivation and explanation thereof.

Lemma 4.4. *There exists a family $(\mathcal{K}_h)_{0<h<1}$ of smooth, non-negative, symmetric functions on $\mathbb{R}^d$ such that $\|\mathcal{K}_h\|_{L^1(\mathbb{R}^d)} \sim |\log h|$ as $h \to 0$, and satisfying*

$$|x||\nabla \mathcal{K}_h(x)| \le C\mathcal{K}_h(x), \tag{10}$$

such that the following holds. Let $(n_k)_k$ be a uniformly bounded sequence in $L^q((0,T)\times\mathbb{R}^d)$, for some $1 \le q < \infty$. If $(\partial_t n_k)_k$ is uniformly bounded in $L^s(0,T;W^{-1,s}(\mathbb{R}^d))$ with $s \ge 1$ and

$$\limsup_k \left(\frac{1}{\|\mathcal{K}_h\|_{L^1(\mathbb{R}^d)}} \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)|n_k(x) - n_k(y)|^q \,\mathrm{d}x\,\mathrm{d}y \right) \longrightarrow 0,$$

as $h \to 0$, then $(n_k)_k$ is compact in $L^q_{loc}((0,T)\times\mathbb{R}^d)$. Conversely, if $(n_k)_k$ is compact in $L^q_{loc}((0,T)\times\mathbb{R}^d)$, then the above quantity vanishes as $h \to 0$.

We refer the reader to the Appendix, where we provide further details of a possible choice for the kernels $\mathcal{K}_h$.
The above lemma is the cornerstone of our compactness argument for the individual species and the total population. However, rather than directly investigating the quantity introduced in the above lemma, let us first consider a weighted version. The weights shall be chosen specifically to fit the problem at hand, as described below.

4.2 Definition of the Weights. We define the weights v_k as solutions of the transport equation

$$\begin{cases} \dfrac{\partial v_k}{\partial t} - \nabla v_k \cdot \nabla W_k &= -\lambda B_k v_k, \\ v_k(0,x) &= 1, \end{cases} \tag{11}$$

where $B_k = \mathrm{M}|\nabla^2 W_k|$. Here λ is some non-negative constant which will be fixed later on. By M we denote the maximal operator, defined by

$$\mathrm{M}f(x) = \sup_{0<\epsilon\leq 1} \frac{1}{|B_0(\epsilon)|} \int_{B_0(\epsilon)} f(x+z)\, \mathrm{d}z,$$

where $B_0(\epsilon)$ denotes the Euclidean ball of radius ϵ centred at zero. Recall that for any $p>1$, the maximal function is a bounded operator on L^p, and we have the following inequality (see, *e.g.*, [27])

$$|\Phi(x) - \Phi(y)| \leq C|x-y|(\mathrm{M}|\nabla\Phi|(x) + \mathrm{M}|\nabla\Phi|(y)),$$

for any $\Phi \in W^{1,1}(\mathbb{R}^d)$. Note that, due to the estimates in Lemma 3.2 and Eq. (3b), we have that B_k, defined in Eq. (11), is uniformly bounded in $L^2((0,T)\times\mathbb{R}^d)$. This allows us to deduce the following properties of the weight v_k.

Proposition 4.5. *Let us assume that ∇W_k is given and that it is bounded in $L^2_{loc}((0,T)\times\mathbb{R}^d) \cap L^\infty(0,T;H^1(\mathbb{R}^d))$ uniformly in k. Thus, there exists a unique solution to Eq. (11). Moreover, we have*

(i) $0 \leq v_k(t,x) \leq 1$, *for almost every $(t,x) \in (0,T)\times\mathbb{R}^d$.*

(ii) *If we assume moreover that $(n_k^{(i)}, W_k)$ satisfies System (3) and $n_k^{(i)}$ is uniformly bounded in $L^2((0,T)\times\mathbb{R}^d)$, there exists $C>0$, such that*

$$\int_{\mathbb{R}^d} n_k^{(i)} |\log v_k| \,\mathrm{d}x \leq C\lambda. \tag{12}$$

Proof.
(i) Since $\nabla^2 W_k \in L^2((0,T)\times\mathbb{R}^d)$, we have that $B_k \in L^2((0,T)\times\mathbb{R}^d)$. Since also $\nabla W_k \in L^2_{loc}((0,T)\times\mathbb{R}^d) \cap L^\infty(0,T;H^1(\mathbb{R}^d))$, by standard theory of renormalised solutions to the transport equations, *cf.* [15], we may construct a non-negative solution to Eq. (11). Moreover, since B_k is non-negative, we clearly have that $v_k \leq 1$, since it is satisfied initially.
(ii) From part (i), we have $|\log v_k| = -\log v_k$. By renormalisation of Eq. (11), we have

$$\frac{\partial}{\partial t}|\log v_k| - \nabla W_k \cdot \nabla |\log v_k| = \lambda B_k.$$

Therefore, using also the continuity equations, Eqs. (3), we get

$$\frac{\partial}{\partial t}\left(n_k^{(i)}|\log v_k|\right) - \nabla\cdot\left(n_k^{(i)}|\log v_k|\nabla W_k\right) = n_k^{(i)}|\log v_k| G^{(i)}(p_k) + \lambda n_k^{(i)} B_k.$$

Upon integrating in space and using the assumptions on $G^{(i)}$, *cf.* Eq. (8), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} n_k^{(i)}|\log v_k|\,\mathrm{d}x \leq G^{(i)}(0)\int_{\mathbb{R}^d} n_k^{(i)}|\log v_k|\,\mathrm{d}x + \lambda \int_{\mathbb{R}^d} n_k^{(i)} B_k\,\mathrm{d}x.$$

Using Gronwall's lemma, we obtain

$$\int_{\mathbb{R}^d} n_k^{(i)}|\log v_k|(T,x)\,\mathrm{d}x \leq \lambda e^{G^{(i)}(0)T}\int_0^T\int_{\mathbb{R}^d} n_k^{(i)} B_k\,\mathrm{d}x\,\mathrm{d}t.$$

Finally, since B_k and $n_k^{(i)}$ are uniformly bounded in $L^2((0,T)\times\mathbb{R}^d)$, we conclude using the Cauchy-Schwarz inequality. Note that this part of the proposition is also true for the total population n_k, replacing "$G^{(i)}(0)$" with "$\max\limits_{i=1,2} G^{(i)}(0)$" in the proof. $\square$

4.3 Propagation of Regularity for the Transport Equation. Here we carry out a preliminary calculation, which will be used in the following subsection. We first compute the equation satisfied by the difference $\left|n_k^{(i)}(x)-n_k^{(i)}(y)\right|$. Using the equation for $n_k^{(i)}$, we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(n_k^{(i)}(x)-n_k^{(i)}(y)\Big)+\nabla_x\cdot\Big(-\nabla W_k(x)\Big(n_k^{(i)}(x)-n_k^{(i)}(y)\Big)\Big)+\nabla_y\cdot\Big(-\nabla W_k(y)\Big(n_k^{(i)}(x)-n_k^{(i)}(y)\Big)\Big)\\
&=-\frac{1}{2}(\Delta W_k(x)+\Delta W_k(y))\Big(n_k^{(i)}(x)-n_k^{(i)}(y)\Big)+\frac{1}{2}(\Delta W_k(x)-\Delta W_k(y))\Big(n_k^{(i)}(x)+n_k^{(i)}(y)\Big)\\
&\quad+\Big(n_k^{(i)}(x)G^{(i)}(p_k(x))-n_k^{(i)}(y)G^{(i)}(p_k(y))\Big).
\end{aligned}$$

Upon multiplying by $\sigma^{(i)}:=\operatorname{sign}\Big(n_k^{(i)}(x)-n_k^{(i)}(y)\Big)$, we deduce

(13)
$$\begin{aligned}
&\frac{\partial}{\partial t}\Big|n_k^{(i)}(x)-n_k^{(i)}(y)\Big|+\nabla_x\cdot\Big(-\nabla W_k(x)\Big|n_k^{(i)}(x)-n_k^{(i)}(y)\Big|\Big)+\nabla_y\cdot\Big(-\nabla W_k(y)\Big|n_k^{(i)}(x)-n_k^{(i)}(y)\Big|\Big)\\
&=-\frac{1}{2}(\Delta W_k(x)+\Delta W_k(y))\Big|n_k^{(i)}(x)-n_k^{(i)}(y)\Big|+\frac{1}{2}(\Delta W_k(x)-\Delta W_k(y))\Big(n_k^{(i)}(x)+n_k^{(i)}(y)\Big)\sigma^{(i)}\\
&\quad+\Big(n_k^{(i)}(x)G^{(i)}(p_k(x))-n_k^{(i)}(y)G^{(i)}(p_k(y))\Big)\sigma^{(i)}.
\end{aligned}$$

Similarly, with $\sigma:=\operatorname{sign}(n_k(x)-n_k(y))$, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}|n_k(x)-n_k(y)|+\nabla_x\cdot(-\nabla W_k(x)|n_k(x)-n_k(y)|)+\nabla_y\cdot(-\nabla W_k(y)|n_k(x)-n_k(y)|)\\
&=\frac{1}{2}(\Delta W_k(x)+\Delta W_k(y))|n_k(x)-n_k(y)|+\frac{1}{2}(\Delta W_k(x)-\Delta W_k(y))(n_k(x)+n_k(y))\sigma\\
&\quad+\Big(n_k^{(1)}(x)G^{(1)}(p_k(x))-n_k^{(1)}(y)G^{(1)}(p_k(y))+n_k^{(2)}(x)G^{(2)}(p_k(x))-n_k^{(2)}(y)G^{(2)}(p_k(y))\Big)\sigma,
\end{aligned}$$

for the total population. The above computations can be made rigorous, for a fixed k, using the renormalisation technique of DiPerna & Lions, *cf.* [15].

4.4 Calculation towards Gronwall's Lemma For $i=1,2$, we now introduce

$$\mathcal{R}_h^{(i)}(t):=\int_{\mathbb{R}^{2d}}\mathcal{K}_h(x-y)\Big|n_k^{(i)}(x)-n_k^{(i)}(y)\Big|(v_k(x)+v_k(y))\,\mathrm{d}x\,\mathrm{d}y,$$

where the weights v_k satisfy Eq. (11). Similarly, for the total population $n_k=n_k^{(1)}+n_k^{(2)}$, we define

$$\mathcal{R}_h^{(tot)}(t):=\int_{\mathbb{R}^{2d}}\mathcal{K}_h(x-y)|n_k(x)-n_k(y)|(v_k(x)+v_k(y))\,\mathrm{d}x\,\mathrm{d}y.$$

Using Eq. (13) we have, for $i = 1, 2$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}^{(i)}(t) = 2\int_{\mathbb{R}^{2d}} &\mathcal{K}_h \Big| n_k^{(i)}(x) - n_k^{(i)}(y)\Big| \frac{\partial v_k}{\partial t}(x)\,\mathrm{d}x\,\mathrm{d}y \\
&- 2\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)\Delta W_k(x)\Big|n_k^{(i)}(x) - n_k^{(i)}(y)\Big|(v_k(x)+v_k(y))\,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)(\Delta W_k(x) - \Delta W_k(y))\Big(n_k^{(i)}(x) + n_k^{(i)}(y)\Big)\sigma^{(i)} v_k(x)\,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)\Big(n_k^{(i)}(x)G^{(i)}(p_k(x)) - n_k^{(i)}(y)G^{(i)}(p_k(y))\Big)\sigma^{(i)}(v_k(x)+v_k(y))\,\mathrm{d}x\,\mathrm{d}y \\
&- \int_{\mathbb{R}^{2d}} \nabla_x \mathcal{K}_h(x-y)\cdot(\nabla W_k(x) - \nabla W_k(y))\Big|n_k^{(i)}(x) - n_k^{(i)}(y)\Big|(v_k(x)+v_k(y))\,\mathrm{d}x\,\mathrm{d}y \\
&- \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)(\nabla W_k(x)\cdot\nabla v_k(x) + \nabla W_k(y)\cdot\nabla v_k(y))\Big|n_k^{(i)}(x) - n_k^{(i)}(y)\Big|\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Upon integrating by parts and using symmetry of the kernel $\mathcal{K}_h$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}^{(i)}(t) = -\int_{\mathbb{R}^{2d}} &\nabla_x \mathcal{K}_h(x-y)\cdot(\nabla W_k(x) - \nabla W_k(y))\Big|n_k^{(i)}(x) - n_k^{(i)}(y)\Big|(v_k(x)+v_k(y))\,\mathrm{d}x\,\mathrm{d}y \\
&+ 2\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)\Big|n_k^{(i)}(x) - n_k^{(i)}(y)\Big|\left(\frac{\partial v_k}{\partial t}(x) - \nabla W_k(x)\cdot\nabla v_k(x)\right)\mathrm{d}x\,\mathrm{d}y \\
&+ 2\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)(\Delta W_k(x) - \Delta W_k(y)) n_k^{(i)}(x)\sigma^{(i)} v_k(x)\,\mathrm{d}x\,\mathrm{d}y \\
&- 2\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)\Delta W_k(x)\Big|n_k^{(i)}(x) - n_k^{(i)}(y)\Big| v_k(x)\,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)\Big(n_k^{(i)}(x)G^{(i)}(p_k(x)) - n_k^{(i)}(y)G^{(i)}(p_k(y))\Big)\sigma^{(i)}(v_k(x)+v_k(y))\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Using Brinkman's law and the a priori bounds on p_k and W_k, we therefore obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}^{(i)}(t) \le \mathcal{A}_1^{(i)} + \mathcal{A}_2^{(i)} + \mathcal{A}_3^{(i)} + \nu^{-1} C\mathcal{R}^{(i)}(t) + \mathrm{React}^{(i)}, \tag{14}$$

where

$$\begin{aligned}
\mathcal{A}_1^{(i)} &= -\int_{\mathbb{R}^{2d}} \nabla\mathcal{K}_h(x-y)\cdot(\nabla W_k(x) - \nabla W_k(y))\Big|n_k^{(i)}(x) - n_k^{(i)}(y)\Big|(v_k(x)+v_k(y))\,\mathrm{d}x\,\mathrm{d}y, \\
\mathcal{A}_2^{(i)} &= 2\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)\Big|n_k^{(i)}(x) - n_k^{(i)}(y)\Big|\left(\frac{\partial v_k}{\partial t}(y) - \nabla W_k(y)\cdot\nabla v_k(y)\right)\mathrm{d}x\,\mathrm{d}y, \\
\mathcal{A}_3^{(i)} &= 2\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)(\Delta W_k(x) - \Delta W_k(y)) n_k^{(i)}(x)\sigma^{(i)} v_k(x)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

as well as

$$\mathrm{React}^{(i)} = \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)\Big(n_k^{(i)}(x)G^{(i)}(p_k(x)) - n_k^{(i)}(y)G^{(i)}(p_k(y))\Big)\sigma^{(i)}(v_k(x)+v_k(y))\,\mathrm{d}x\,\mathrm{d}y.$$

Similarly, we obtain for the total population,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}^{(tot)}(t) \le \mathcal{A}_1^{(tot)} + \mathcal{A}_2^{(tot)} + \mathcal{A}_3^{(tot)} + \nu^{-1} C\mathcal{R}^{(tot)}(t) + \mathrm{React}^{(tot)}, \tag{15}$$

where

$$\mathcal{A}_1^{(tot)} = -\int_{\mathbb{R}^{2d}} \nabla\mathcal{K}_h(x-y)\cdot(\nabla W_k(x)-\nabla W_k(y))|n_k(x)-n_k(y)|(v_k(x)+v_k(y))\,\mathrm{d}x\,\mathrm{d}y,$$

$$\mathcal{A}_2^{(tot)} = 2\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)|n_k(x)-n_k(y)|\left(\frac{\partial v_k}{\partial t}(y)-\nabla W_k(y)\cdot\nabla v_k(y)\right)\mathrm{d}x\,\mathrm{d}y,$$

$$\mathcal{A}_3^{(tot)} = 2\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)(\Delta W_k(x)-\Delta W_k(y))n_k(x)\sigma v_k(x)\,\mathrm{d}x\,\mathrm{d}y,$$

and

$$\mathrm{React}^{(tot)} = \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)\sum_{i=1}^{2}\left(n_k^{(i)}(x)G^{(i)}(p_k(x))-n_k^{(i)}(y)G^{(i)}(p_k(y))\right)\sigma(v_k(x)+v_k(y))\,\mathrm{d}x\,\mathrm{d}y.$$

In what follows, we consider the quantity

$$\mathcal{R}(t) := \mathcal{R}^{(1)}(t)+\mathcal{R}^{(2)}(t)+\mathcal{R}^{(tot)}(t).$$

As it turns out, this quantity allows for convenient cancellations in the reaction terms, and in the terms "$\mathcal{A}_3$". This strategy is reminiscent of the BV-norm propagation argument employed in the one-dimensional case, *cf.* [14]. Accordingly, we define the following notation

$$\mathcal{A}_j = \mathcal{A}_j^{(1)}+\mathcal{A}_j^{(2)}+\mathcal{A}_j^{(tot)},$$

for $j=1,2,3$, and

$$\mathrm{React} = \mathrm{React}^{(1)}+\mathrm{React}^{(2)}+\mathrm{React}^{(tot)}.$$

Combining Eq. (14) and Eq. (15) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}(t) \le \mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3+\nu^{-1}C\mathcal{R}(t)+\mathrm{React}. \tag{16}$$

The terms $\mathcal{A}_1$ and $\mathcal{A}_2$ are (almost) the same as in previous works, *cf.* [7, 8, 28], and can be handled using only the uniform bound on the velocity field $-\nabla W_k$. We therefore begin by estimating the reaction terms and the term $\mathcal{A}_3$.

4.5 Treating the Reaction Terms. First let us address the contribution coming from the reaction terms, $\mathrm{React}^{(tot)}$. Recalling the notation $\sigma^{(i)} = \mathrm{sign}(n_k^{(i)}(x)-n_k^{(i)}(y))$, we obtain for the two individual species

$$\begin{aligned}\mathrm{react}^{(i)} &:= \left(n_k^{(i)}(x)G^{(i)}(p_k(x))-n_k^{(i)}(y)G^{(i)}(p_k(y)\right)\sigma^{(i)}\\ &= G^{(i)}(p_k(x))\left|n_k^{(i)}(x)-n_k^{(i)}(y)\right|+\left(G^{(i)}(p_k(x))-G^{(i)}(p_k(y))\right)n_k^{(i)}(y)\sigma^{(i)}\\ &\le \|G^{(i)}\|_\infty\left|n_k^{(i)}(x)-n_k^{(i)}(y)\right|+\left|G^{(i)}(p_k(x))-G^{(i)}(p_k(y))\right|n_k^{(i)}(y),\end{aligned}$$

having used the fact that $|\sigma^{(i)}|\le 1$.

Similarly, an expression for the sum, $n_k = n_k^{(1)} + n_k^{(2)}$, may be obtained, *i.e.*,

$$\begin{aligned}\mathrm{react}^{(tot)} &:= \Big(n_k^{(1)}(x)G^{(1)}(p_k(x)) - n_k^{(1)}(y)G^{(1)}(p_k(y) + n_k^{(2)}(x)G^{(2)}(p_k(x)) - n_k^{(2)}(y)G^{(2)}(p_k(y)\Big)\sigma \\ &= G^{(1)}(p_k(x))\Big(n_k^{(1)}(x) - n_k^{(1)}(y)\Big)\sigma + \Big(G^{(1)}(p_k(x)) - G^{(1)}(p_k(y))\Big)n_k^{(1)}(y)\sigma \\ &\quad + G^{(2)}(p_k(x))\Big(n_k^{(2)}(x) - n_k^{(2)}(y)\Big)\sigma + \Big(G^{(2)}(p_k(x)) - G^{(2)}(p_k(y))\Big)n_k^{(2)}(y)\sigma \\ &\leq \|G^{(1)}\|_\infty\Big|n_k^{(1)}(x) - n_k^{(1)}(y)\Big| + \|G^{(2)}\|_\infty\Big|n_k^{(2)}(x) - n_k^{(2)}(y)\Big| \\ &\quad + \Big(G^{(1)}(p_k(x)) - G^{(1)}(p_k(y))\Big)n_k^{(1)}(y)\sigma + \Big(G^{(2)}(p_k(x)) - G^{(2)}(p_k(y))\Big)n_k^{(2)}(y)\sigma.\end{aligned}$$

Using the fact that

$$\sigma := \mathrm{sign}(n_k(x) - n_k(y)) = \mathrm{sign}(p_k(x) - p_k(y)),$$

and that the functions $G^{(i)}$ are decreasing, it is readily verified that adding up all three contributions yields

$$\mathrm{react}^{(1)} + \mathrm{react}^{(2)} + \mathrm{react}^{(tot)} \leq C\Big(\Big|n_k^{(1)}(x) - n_k^{(1)}(y)\Big| + \Big|n_k^{(2)}(x) - n_k^{(2)}(y)\Big|\Big),$$

for some constant $C > 0$, independent of k. It follows that the whole term can be estimated as

$$\mathrm{React} \leq C\mathcal{R}(t).$$

4.6 Treatment of the Highest Order Terms. We now address the contribution of the term $\mathcal{A}_3$. Recall that

$$\begin{aligned}\mathcal{A}_3 = &\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)(\Delta W_k(x) - \Delta W_k(y)) \sum_{i=1}^{2} \sigma^{(i)} n_k^{(i)}(x) v_k(x)\,\mathrm{d}x\,\mathrm{d}y \\ &+ \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)(\Delta W_k(x) - \Delta W_k(y))\sigma n_k(x) v_k(x)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Next, notice that

$$\begin{aligned}a_3^{(i)} &:= (\Delta W_k(x) - \Delta W_k(y))\sigma^{(i)} n_k^{(i)}(x) \\ &= \nu^{-1}(W_k(x) - p_k(x) - W_k(y) + p_k(y))\sigma^{(i)} n_k^{(i)}(x) \\ &\leq \nu^{-1}|W_k(x) - W_k(y)|\Big\|n_k^{(i)}\Big\|_{L^\infty} + \nu^{-1}|p_k(x) - p_k(y)|n_k^{(i)}(x),\end{aligned}$$

having used the fact that $\sigma^{(i)} \leq 1$, as before. Finally, the term stemming from the joint population reads

$$\begin{aligned}a_3^{(tot)} &:= (\Delta W_k(x) - \Delta W_k(y))\sigma n_k(x) \\ &= \nu^{-1}(W_k(x) - p_k(x) - W_k(y) + p_k(y))\sigma n_k(x) \\ &= \nu^{-1}\sigma(W_k(x) - W_k(y))n_k(x) - \nu^{-1}|p_k(x) - p_k(y)|n_k(x) \\ &\leq \nu^{-1}|W_k(x) - W_k(y)|\|n_k\|_{L^\infty} - \nu^{-1}|p_k(x) - p_k(y)|n_k(x).\end{aligned}$$

Again, using the a-priori bounds, it is immediate that adding up all three terms yields

$$a_3^{(1)} + a_3^{(2)} + a_3^{(tot)} \leq \nu^{-1}C|W_k(x) - W_k(y)|,$$

due to the cancellation $n_k^{(1)} + n_k^{(2)} - n_k = 0$. Therefore we obtain the following bound

$$\mathcal{A}_3 \leq \nu^{-1}C\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)|W_k(x) - W_k(y)|v_k(x)\,\mathrm{d}x\,\mathrm{d}y. \tag{17}$$

This concludes the estimate of the highest order term for now. Let us just remark at this stage that the quantity on the right-hand side vanishes, as $h \to 0$, due to local compactness of W_k and the "converse" statement of Lemma 4.4.

4.7 Estimates of $\mathcal{A}_1$ and $\mathcal{A}_2$. For the readers' convenience we recall how to estimate the remaining terms in Eq. (16).
First, we make use of the following inequality

$$|\nabla W_k(x) - \nabla W_k(y)| \le C|x-y|\big(\mathrm{D}_{|x-y|}\nabla W_k(x) + \mathrm{D}_{|x-y|}\nabla W_k(y)\big),$$

where

$$\mathrm{D}_\delta \nabla W_k(x) = \frac{1}{\delta}\int_{|z|\le\delta} \frac{\left|\nabla^2 W_k(x+z)\right|}{|z|^{d-1}}\,\mathrm{d}z,$$

see, for example, [19, Lemma 3.1]. We recall also that $\mathrm{D}_\delta\nabla W_k \le \mathrm{M}\left|\nabla^2 W_k\right|$, Then, using the inequality in Eq. (10) and the symmetry of $\mathcal{K}_h$ we get

$$\begin{aligned}\mathcal{A}_1 \le C\int_{\mathbb{R}^{2d}} &|x-y||\nabla\mathcal{K}_h(x-y)|\big(\mathrm{D}_{|x-y|}\nabla W_k(x) + \mathrm{D}_{|x-y|}\nabla W_k(y)\big)\\ &\qquad\left(\sum_{i=1}^{2}\left|n_k^{(i)}(x) - n_k^{(i)}(y)\right| + |n_k(x) - n_k(y)|\right)(v_k(x)+v_k(y))\,\mathrm{d}x\,\mathrm{d}y\\ \le C\int_{\mathbb{R}^{2d}} &\mathcal{K}_h(x-y)\big(\mathrm{D}_{|x-y|}\nabla W_k(x) + \mathrm{D}_{|x-y|}\nabla W_k(y)\big)\\ &\qquad\left(\sum_{i=1}^{2}\left|n_k^{(i)}(x) - n_k^{(i)}(y)\right| + |n_k(x) - n_k(y)|\right)v_k(y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Using that

$$\mathrm{D}_{|x-y|}\nabla W_k(x) + \mathrm{D}_{|x-y|}\nabla W_k(y) = \mathrm{D}_{|x-y|}\nabla W_k(x) - \mathrm{D}_{|x-y|}\nabla W_k(y) + 2\mathrm{D}_{|x-y|}\nabla W_k(y),$$

and changing the variables $z = x - y$, we may apply the Cauchy-Schwarz inequality and use the uniform L^2-bounds on $n_k^{(i)}$ and n_k to deduce

$$\begin{aligned}\mathcal{A}_1 \le\, &C\int_{\mathbb{R}^d}\mathcal{K}_h(z)\left\|\mathrm{D}_{|z|}\nabla W_k(\cdot) - \mathrm{D}_{|z|}\nabla W_k(\cdot+z)\right\|_{L^2}\mathrm{d}z\\ &+ C\int_{\mathbb{R}^{2d}}\mathcal{K}_h(x-y)\mathrm{D}_{|x-y|}\nabla W_k(y)\sum_{i=1}^{2}\left|n_k^{(i)}(x) - n_k^{(i)}(y)\right|v_k(y)\,\mathrm{d}x\,\mathrm{d}y\\ &+ C\int_{\mathbb{R}^{2d}}\mathcal{K}_h(x-y)\mathrm{D}_{|x-y|}\nabla W_k(y)|n_k(x) - n_k(y)|v_k(y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We may bound $\mathrm{D}_{|x-y|}\nabla W_k$ by the maximal operator $\mathrm{M}\left|\nabla^2 W_k\right|$, thus

$$\begin{aligned}\mathcal{A}_1 \le\, &C\int_{\mathbb{R}^d}\mathcal{K}_h(z)\left\|\mathrm{D}_{|z|}\nabla W_k(\cdot) - \mathrm{D}_{|z|}\nabla W_k(\cdot+z)\right\|_{L^2}\mathrm{d}z\\ &+ C\int_{\mathbb{R}^{2d}}\mathcal{K}_h(x-y)\mathrm{M}\left|\nabla^2 W_k(y)\right|\sum_{i=1}^{2}\left|n_k^{(i)}(x) - n_k^{(i)}(y)\right|v_k(y)\,\mathrm{d}x\,\mathrm{d}y\\ &+ C\int_{\mathbb{R}^{2d}}\mathcal{K}_h(x-y)\mathrm{M}\left|\nabla^2 W_k(y)\right||n_k(x) - n_k(y)|v_k(y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{18}$$

The last two terms on the right-hand side of inequality (18) will be controlled by the term $\mathcal{A}_2$ and the equation for the weight v_k.

From Eq. (11), we have

$$\mathcal{A}_2 = \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y) \sum_{i=1}^{2} \Big| n_k^{(i)}(x) - n_k^{(i)}(y) \Big| (-2\lambda B_k(y)) v_k(y) \, \mathrm{d}x \, \mathrm{d}y$$
$$+ \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y) |n_k(x) - n_k(y)| (-2\lambda B_k(y)) v_k(y) \, \mathrm{d}x \, \mathrm{d}y.$$

Therefore, combining the latter equality with Eq. (18), we deduce

$$\mathcal{A}_1 + \mathcal{A}_2 \leq C \int_{\mathbb{R}^d} \mathcal{K}_h(z) \big\| \mathrm{D}_{|z|} \nabla W_k(\cdot) - \mathrm{D}_{|z|} \nabla W_k(\cdot + z) \big\|_{L^2} \mathrm{d}z$$
$$+ \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y) \sum_{i=1}^{2} \Big| n_k^{(i)}(x) - n_k^{(i)}(y) \Big| v_k(y) \big(C\mathrm{M}|\nabla^2[0.5em] W_k(y)| - 2\lambda B_k(y) \big) \, \mathrm{d}x \, \mathrm{d}y$$
$$+ \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y) |n_k(x) - n_k(y)| v_k(y) \big(C\mathrm{M}|\nabla^2 W_k(y)| - 2\lambda B_k(y) \big) \, \mathrm{d}x \, \mathrm{d}y.$$

From the choice of B_k in Eq. (11), we can find λ large enough such that

$$\mathcal{A}_1 + \mathcal{A}_2 \leq C \int_{\mathbb{R}^d} \mathcal{K}_h(z) \big\| \mathrm{D}_{|z|} \nabla W_k(\cdot) - \mathrm{D}_{|z|} \nabla W_k(\cdot + z) \big\|_{L^2} \mathrm{d}z.$$

We shall now employ the following lemma, which we state without proof, referring the reader to [8].

Lemma 4.6 (Lemma 6.3 in [8]). *There exists a constant, $C > 0$, such that for any $u \in H^1(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} \mathcal{K}_h(z) \big\| \mathrm{D}_{|z|} u(\cdot) - \mathrm{D}_{|z|} u(\cdot + z) \big\|_{L^2(\mathbb{R}^d)} \mathrm{d}z \leq C |\log h|^{1/2} \|u\|_{H^1(\mathbb{R}^d)}.$$

In our case we take $u = \nabla W_k$, which indeed belongs to H^1, uniformly. Thus, we have the estimate

$$\mathcal{A}_1 + \mathcal{A}_2 \leq C |\log h|^{1/2} \|W_k\|_{W^{2,2}(\mathbb{R}^d)} \leq C |\log h|^{1/2}. \tag{19}$$

4.8 Conclusions from the Estimates. Departing from Eq. (16), we obtain, in conjunction with Estimate (17) (coming from the term $\mathcal{A}_2$) and Estimate (19) (coming from the terms $\mathcal{A}_1$ and $\mathcal{A}_3$),

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{R}_h(t) \leq C \, \mathcal{R}_h(t) + C |\log h|^{1/2} + \nu^{-1} C \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y) |W_k(x) - W_k(y)| v_k(x) \, \mathrm{d}x \, \mathrm{d}y.$$

Then, integrating in time, we obtain, for all $t \in [0, T]$,

$$\begin{aligned} e^{-Ct} \mathcal{R}_h(t) \leq \mathcal{R}_h(0) &+ C_T |\log h|^{1/2} \\ &+ \nu^{-1} C_T \int_0^T \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y) |W_k(x) - W_k(y)| v_k(x) \, \mathrm{d}x \, \mathrm{d}y \ \mathrm{d}t. \end{aligned} \tag{20}$$

Regarding the last term we observe that

$$\nu^{-1} C_T \int_0^T \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y) |W_k(x) - W_k(y)| v_k(x) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t$$
$$\leq \nu^{-1} C_T \int_0^T \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y) |W_k(x) - W_k(y)| \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t.$$

Since $(W_k)_k$ is locally compact in $L^1((0,T) \times \mathbb{R}^d)$, the right-most integral vanishes in the limit $h \to 0$, due to the "converse" statement of Lemma 4.4.

4.9 Removing the Weights and Application of the Compactness Argument. Let $0 < \eta < 1$. We define

$$\omega_\eta = \{x \in \mathbb{R}^d : v_k(x) \leq \eta\},$$

and denote by ω_η^c its complement in $\mathbb{R}^d$. We then split integration over the domain $\mathbb{R}^{2d}$ as follows

$$\begin{aligned}\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)|n_k(x)-n_k(y)|\,\mathrm{d}x\,\mathrm{d}y &= \int_{\{x\in\omega_\eta^c\}\cup\{y\in\omega_\eta^c\}} \mathcal{K}_h(x-y)|n_k(x)-n_k(y)|\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{\{x\in\omega_\eta\}\cap\{y\in\omega_\eta\}} \mathcal{K}_h(x-y)|n_k(x)-n_k(y)|\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{21}$$

These integrals can now be estimated separately, in turn we have

$$\int_{\{x\in\omega_\eta^c\}\cup\{y\in\omega_\eta^c\}} \mathcal{K}_h(x-y)|n_k(x)-n_k(y)|\,\mathrm{d}x\,\mathrm{d}y \leq \frac{2}{\eta}\mathcal{R}_h(t),$$

and

$$\begin{aligned}\int_{\{x\in\omega_\eta\}\cap\{y\in\omega_\eta\}} \mathcal{K}_h(x-y)|n_k(x)-n_k(y)|\,\mathrm{d}x\,\mathrm{d}y &\leq 2\int_{\{x\in\omega_\eta\}\cap\{y\in\omega_\eta\}} \mathcal{K}_h(x-y)n_k(x)\,\mathrm{d}x\,\mathrm{d}y \\ &\leq 2\int_{\mathbb{R}^d} \mathcal{K}_h(z)\,\mathrm{d}z \int_{\{x\in\omega_\eta\}} n_k(x)\,\mathrm{d}x \\ &\leq C|\log h| \int_{\{x\in\omega_\eta\}} n_k(x)\,\mathrm{d}x,\end{aligned}$$

where we used the symmetry of $\mathcal{K}_h$ and the fact that $\|\mathcal{K}_h\|_{L^1} \sim |\log h|$. In order to treat the last integral we observe that

$$\int_{\{x\in\omega_\eta\}} n_k(x)\,\mathrm{d}x \leq \int_{\{x\in\omega_\eta\}} n_k(x)\frac{|\log v_k(x)|}{|\log \eta|}\,\mathrm{d}x \leq C\frac{1}{|\log \eta|},$$

where the last inequality holds due to the uniform bound on $n_k|\log v_k|$, *cf.* Eq. (12), Proposition 4.5, and the fact that, for $\eta < 1$, $|\log v_k(x)| \geq |\log \eta|$, whenever $x \in \omega_\eta$. Substituting these estimates into Eq. (21), we arrive at

$$\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)|n_k(x)-n_k(y)|\,\mathrm{d}x\,\mathrm{d}y \leq \frac{2}{\eta}\mathcal{R}_h(t) + \frac{C|\log h|}{|\log \eta|}.$$

Therefore, from Eq. (20), we deduce

$$\begin{aligned}\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)|n_k(x)-n_k(y)|\,\mathrm{d}x\,\mathrm{d}y &\leq \frac{C_T}{\eta}\Big(\mathcal{R}_h(0) + |\log h|^{1/2}\Big) \\ &\quad + \frac{C_T}{\nu\eta}\int_0^T\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)|W_k(x)-W_k(y)|\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{C|\log h|}{|\log \eta|}.\end{aligned}$$

Since we have $\|\mathcal{K}_h\|_{L^1} \sim |\log h|$, we obtain

$$\begin{aligned}\frac{1}{\|\mathcal{K}_h\|_{L^1}} \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)|n_k(x)-n_k(y)| \,\mathrm{d}x \,\mathrm{d}y \\ &\le \frac{C_T}{\eta}\Big(\mathcal{R}_h(0)|\log h|^{-1} + |\log h|^{-1/2}\Big) \\ &\quad + \frac{C_T}{\nu\eta|\log h|}\int_0^T\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)|W_k(x)-W_k(y)| \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{C}{|\log \eta|}.\end{aligned}$$

Choosing $\eta = |\log h|^{-1/4}$, we have $\eta \to 0$ as $h \to 0$. Then

$$\begin{aligned}\frac{1}{\|\mathcal{K}_h\|_{L^1}} \int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)|n_k(x)-n_k(y)| \,\mathrm{d}x \,\mathrm{d}y \\ &\le C_T\Big(\mathcal{R}_h(0)|\log h|^{-3/4} + |\log h|^{-1/4}\Big) \\ &\quad + \frac{C_T}{|\log h|^{3/4}}\int_0^T\int_{\mathbb{R}^{2d}} \mathcal{K}_h(x-y)|W_k(x)-W_k(y)| \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{C}{|\log|\log h||}.\end{aligned}$$

Of course, the above calculations are oblivious to replacing n_k with $n_k^{(i)}$. Hence, finally, we obtain the desired compactness of the sequences $(n_k)_k$ and $(n_k^{(i)})_k$, $i=1,2$, as stated in Theorem 4.1, by applying the compactness criterion in Lemma 4.4. Indeed the estimate on the time derivative is a direct consequence of the a priori estimates.

Proof of Corollary 4.2. From Remark 3.4 in conjunction with the fact that $0 \le r_k \le 1$, we infer the boundedness of the sequence $(r_k)_k$ in $L^\infty(0,T; L^1_{loc} \cap L^\infty_{loc}(\mathbb{R}^d))$. Since both $n_k^{(i)}$ and n_k converge not only strongly but also pointwise, we deduce the strong compactness of the sequence $(r_k)_k$ in any $L^p(0,T; L^q_{loc}(\mathbb{R}^d))$, for $1 \le p, q < \infty$. □

5 Strong Compactness of the Pressure

In order to be able to pass to the incompressible limit, stated in Theorem 2.3, we require strong convergence of the pressure sequence $(p_k)_k$. As pointed out before, *cf.* Remark 4.3, we cannot hope to propagate initial compactness for the pressure and therefore the strategy of Section 4 cannot be employed. Accordingly, a different approach has to be found. The main challenge in this endeavour is the absence of any estimates on the derivative of p_k, unlike in the inviscid case, *cf.* [9, 20, 22–24]. To remedy this, we follow the idea of kinetic reformulations developed in [25], and thus investigate possible oscillations of the sequence of pressures. Let us recall the equation satisfied by the pressure, p_k, *i.e.*,

$$\frac{\partial p_k}{\partial t} - \nabla p_k \cdot \nabla W_k = \frac{k-1}{\nu} p_k\Big[W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1-r_k)G^{(2)}(p_k)\Big] = \frac{k-1}{\nu} p_k Q_k. \tag{22}$$

We now introduce a family of functions $\{H_r\}_r$ as follows. Let

$$H_r(W) := H(r,W) := \Big[\mathrm{Id} - \nu r G^{(1)}(\cdot) - \nu(1-r)G^{(2)}(\cdot)\Big]^{-1}(W),$$

where the inverse is to be understood in the p-variable. For a fixed r, we then have that H_r is increasing and has a bounded derivative, *i.e.*, $H_r' < C$, for some $C > 0$. Moreover, notice that for any fixed $r \in [0,1]$ we have $H_r(0) > 0$. We point out that, for the subsequent analysis, it will prove useful to introduce the number

$$p_m := \min_{r\in[0,1]} H(r,0) > 0. \tag{23}$$

The remainder of this section is dedicated to proving the following result.

Lemma 5.1 (Strong convergence of p_k). *Up to a subsequence, p_k converges strongly in $L^1_{loc}\big((0,T)\times\mathbb{R}^d\big)$ towards a function p_∞. Moreover, the limit function is characterised by $p_\infty = H(r_\infty, W_\infty)\mathbb{1}_{\{p_\infty>0\}}$ a.e.. Furthermore, we have that*

$$\Omega(t) = \{p_\infty(\cdot,t) = H(r_\infty, W_\infty(\cdot,t))\} = \mathbb{R}^d\backslash\{p_\infty(\cdot,t)=0\},$$

is the image of $\Omega^0 := \{p_\infty(t=0)>0\}$ by the limiting flow $Y_{(x)}(t)$, defined by

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}Y_{(x)}(t) &= -\nabla W_\infty(t, Y_{(x)}(t)),\\ Y_{(x)}(t=0) &= \mathrm{Id}\,. \end{cases}$$

Finally, we have for all $T>0$,

$$k\int_0^T\int_{\mathbb{R}^d} p_k(t,x)|Q_k(t,x)|\,\mathrm{d}x\,\mathrm{d}t \underset{k\to+\infty}{\longrightarrow} 0. \tag{24}$$

5.1 Proof of the Lemma. Step 1. Representation of Nonlinear Weak Limits. We follow closely the proof in [25], making appropriate adaptations to our current case. To this end, we need a representation of weak limits of p_k which we can obtain thanks to a kinetic representation.

Our first result is to establish the existence of a measurable function $0\le f(t,x)\le 1$ such that, for all smooth functions $S:[0,\infty)\to\mathbb{R}$, we have, up to a subsequence,

$$S(p_k) \underset{k\to+\infty}{\rightharpoonup} S(0)(1-f) + S(H(r_\infty, W_\infty))f, \tag{25}$$

in the weak-$\star$ sense in $L^\infty((0,T)\times\mathbb{R}^d)$ and

$$S(0)(1-f) + S(H(r_\infty,W_\infty))f = S(0) + \int_0^\infty S'(\xi)\chi(\xi)\,\mathrm{d}\xi, \tag{26a}$$

where

$$\chi(t,x;\xi) = f(t,x)\mathbb{1}_{\{0<\xi<H(r_\infty(t,x),W_\infty(t,x))\}}. \tag{26b}$$

In particular, notice that for $S(p)=p$, we find

$$p_\infty = f\ H(r_\infty, W_\infty). \tag{27}$$

In order to prove these results, we define

$$\chi_k(t,x;\xi) = \mathbb{1}_{\{0<\xi<p_k(t,x)\}},$$

and we write

$$S(p_k) - S(0) = \int_0^\infty S'(\xi)\chi_k(t,x;\xi)\,\mathrm{d}\xi. \tag{28}$$

We can extract a subsequence, still denoted $(p_k)_k$, such that $\{\chi_k\}_k$ converges in the weak-$\star$ sense in $L^\infty((0,\infty)\times\mathbb{R}^d)$ towards a function $\chi(t,x;\xi)$ which satisfies $0\le\chi(t,x;\xi)\le 1$.
We define f to be the weak-$\star$ limit of $\mathbb{1}_{\{p_k(t,x)\ge p_m/2\}}$, *i.e.*,

$$f(t,x) := \text{weak-}\star-\lim \mathbb{1}_{\{p_k(t,x)\ge p_m/2\}},$$

where we recall that p_m is defined in Eq. (23).

To prove the convergence in Eq. (25) we first split $S(p_k)$ as follows

$$S(p_k) = S(p_k)\mathbb{1}_{\{p_k<p_m/2\}} + S(p_k)\mathbb{1}_{\{p_k\ge p_m/2\}},$$

and pass to the weak-$\star$ limit in each term separately. Let now $\phi \in C_c^\infty((0,T)\times\mathbb{R}^d)$ be a test function. We then have

(29)

$$\left|\int_0^T\int_{\mathbb{R}^d}\phi S(p_k)\mathbb{1}_{\{p_k<p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\mathbb{R}^d}\phi S(0)(1-f)\,\mathrm{d}x\,\mathrm{d}t\right|$$
$$\leq \left|\int_0^T\int_{\mathbb{R}^d}\phi(S(p_k)-S(0))\mathbb{1}_{\{p_k<p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t\right| + \left|\int_0^T\int_{\mathbb{R}^d}\phi S(0)\big(\mathbb{1}_{\{p_k<p_m/2\}}-(1-f)\big)\,\mathrm{d}x\,\mathrm{d}t\right|,$$

and the last term converges to zero by definition of f. To deal with the other term on the right-hand side of Eq. (29) we proceed as follows.

$$\left|\int_0^T\int_{\mathbb{R}^d}\phi(S(p_k)-S(0))\mathbb{1}_{\{p_k<p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t\right| \leq \|\phi\|_{L^\infty}\|S'\|_{L^\infty}\int_0^T\int_{\mathbb{R}^d}p_k\mathbb{1}_{\{p_k\leq p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t.$$

Notice that $H(r_\infty,W_\infty) > p_m$. Therefore on the set $\{p_k \leq p_m/2\}$ we have

$$p_k \leq H(r_\infty,W_\infty) - p_m/2,$$

and thus

$$H^{-1}_{r_\infty}(p_k) \leq H^{-1}_{r_\infty}(H(r_\infty,W_\infty)-p_m/2) \leq W_\infty - \omega(p_m),$$

for some $\omega(p_m) > 0$. In particular, on the set $\{p_k \leq p_m/2\}$ we have

$$1 \leq \frac{1}{\omega(p_m)}\big(W_\infty - H^{-1}_{r_\infty}(p_k)\big).$$

Therefore

(30)

$$\int_0^T\int_{\mathbb{R}^d}p_k\mathbb{1}_{\{p_k\leq p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t \leq C\int_0^T\int_{\mathbb{R}^d}p_k\big|W_\infty - H^{-1}_{r_\infty}(p_k)\big|\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq C\int_0^T\int_{\mathbb{R}^d}p_k|W_\infty-W_k|\,\mathrm{d}x\,\mathrm{d}t + C\int_0^T\int_{\mathbb{R}^d}p_k\big|W_k - H^{-1}_{r_k}(p_k)\big|\,\mathrm{d}x\,\mathrm{d}t$$
$$+ C\int_0^T\int_{\mathbb{R}^d}p_k\big|H^{-1}_{r_k}(p_k) - H^{-1}_{r_\infty}(p_k)\big|\,\mathrm{d}x\,\mathrm{d}t.$$

Recalling that $p_k \leq P_M$, and using the definition of the function $H_r(\cdot)$, we arrive at

$$\int_0^T\int_{\mathbb{R}^d}p_k\mathbb{1}_{\{p_k\leq p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t \leq C\int_0^T\int_{\mathbb{R}^d}P_H|W_\infty-W_k|\,\mathrm{d}x\,\mathrm{d}t + C\int_0^T\int_{\mathbb{R}^d}p_k|Q_k|\,\mathrm{d}x\,\mathrm{d}t$$
$$+ C\int_0^T\int_{\mathbb{R}^d}P_H\Big(\nu G^{(1)}(0)+\nu G^{(2)}(0)\Big)|r_k-r_\infty|\,\mathrm{d}x\,\mathrm{d}t.$$

Note that the last three integrals vanish in the limit $k\to\infty$. In fact, the first integral converges to zero due to strong convergence of W_k, the second one due to Lemma 3.5, and third integral vanishes due to strong convergence of r_k, *cf.* Corollary 4.2.

Similarly, we have

$$\left|\int_0^T\int_{\mathbb{R}^d}\phi S(p_k)\mathbb{1}_{\{p_k\geq p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\mathbb{R}^d}\phi S(H(r_\infty,W_\infty))f\,\mathrm{d}x\,\mathrm{d}t\right|$$
$$\leq \int_0^T\int_{\mathbb{R}^d}|\phi||S(p_k)-S(H(r_\infty,W_\infty))|\mathbb{1}_{\{p_k\geq p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\left|\int_0^T\int_{\mathbb{R}^d}\phi S(H(r_\infty,W_\infty))\big(f-\mathbb{1}_{\{p_k\geq p_m/2\}}\big)\,\mathrm{d}x\,\mathrm{d}t\right|,$$

and again we need only work on the first term on the right-hand side. We have

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}|\phi||S(p_k)-S(H(r_\infty,W_\infty))|\mathbb{1}_{\{p_k\geq p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leq C\int_0^T\int_{\mathbb{R}^d}|S(p_k)-S(H(r_k,W_k))|\mathbb{1}_{\{p_k\geq p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\quad+C\int_0^T\int_{\mathbb{R}^d}|S(H(r_k,W_k))-S(H(r_k,W_\infty))|\mathbb{1}_{\{p_k\geq p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\quad+C\int_0^T\int_{\mathbb{R}^d}|S(H(r_k,W_\infty))-S(H(r_\infty,W_\infty))|\mathbb{1}_{\{p_k\geq p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{31}
$$

For the first term on the right-hand side of the last inequality we write

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}|S(p_k)-S(H(r_k,W_k))|\mathbb{1}_{\{p_k\geq p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leq\|S'\|_{L^\infty}\int_0^T\int_{\mathbb{R}^d}|p_k-H(r_k,W_k)|\mathbb{1}_{\{p_k\geq p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leq\|S'\|_{L^\infty}\|H'_{r_k}\|_{L^\infty}\int_0^T\int_{\mathbb{R}^d}\big|H^{-1}_{r_k}(p_k)-W_k\big|\mathbb{1}_{\{p_k\geq p_m/2\}}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leq\|S'\|_{L^\infty}\|H'_{r_k}\|_{L^\infty}\frac{2}{p_m}\int_0^T\int_{\mathbb{R}^d}p_k\big|H^{-1}_{r_k}(p_k)-W_k\big|\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad=\|S'\|_{L^\infty}\|H'_{r_k}\|_{L^\infty}\frac{2}{p_m}\int_0^T\int_{\mathbb{R}^d}p_k|Q_k|\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leq\frac{C(T)}{k}\longrightarrow 0,
\end{aligned}\tag{32}
$$

as $k\to\infty$, having applied Lemma 3.5 and used the fact that H'_{r_k} is uniformly bounded due to $0\leq r_k\leq 1$. In a similar fashion the last two terms in Eq. (31) vanish in the limit $k\to\infty$, due to strong convergence of W_k and r_k, respectively.
This concludes the proof of the representation of weak limits, *cf.* Eq. (25). The final statement, Eq. (26b), follows from passing to the limit $k\to\infty$ in the identity

$$
S(0)+\int_0^\infty S'(\xi)\chi_k(t,x;\xi)\,\mathrm{d}\xi=S(p_k)=S(p_k)\mathbb{1}_{\{p_k(t,x)\geq p_m/2\}}+S(p_k)\mathbb{1}_{\{p_k\leq p_m/2\}}.
$$

Using the fact that p_k does not oscillate we deduce

$$
S(0)+\int_0^\infty S'(\xi)\chi(t,x;\xi)\,\mathrm{d}\xi=S(H(r_\infty,W_\infty))f+S(0)(1-f).
$$

Rearranging this expression yields

$$
f(t,x)\int_0^\infty S'(\xi)\mathbb{1}_{\{0\leq\xi\leq H(r_\infty,W_\infty)\}}(\xi)\,\mathrm{d}\xi=\int_0^\infty S'(\xi)\,\mathrm{d}\xi,
$$

whence $\chi(t,x;\xi)=f(t,x)\mathbb{1}_{\{0\leq\xi\leq H(r_\infty,W_\infty)\}}$, as claimed.

5.2 Proof of Lemma. Step 2. Equation satisfied by χ_k. Let $S\in C^2([0,\infty))$. Upon rewritting Eq. (22) in divergence form and multiplying by $S'(p_k)$ we obtain

$$
\frac{\partial}{\partial t}S(p_k)-\nabla\cdot(S(p_k)\nabla W_k)+S'(p_k)\frac{W_k-p_k}{\nu}=\frac{k-1}{\nu}p_kQ_kS'(p_k)=\int_0^\infty S'(\xi)\mu_k(\mathrm{d}\xi),
$$

where the measure μ_k is defined via

$$
\begin{aligned}
\mu_k(t,x;\xi) &:= \frac{k-1}{\nu} p_k Q_k \delta_{\{\xi=p_k\}} \\
&= \frac{k-1}{\nu} p_k \Big[W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1-r_k) G^{(2)}(p_k) \Big] \delta_{\{\xi=p_k\}}.
\end{aligned} \tag{33}
$$

Then, using Eq. (28) and the fact that $S(p_k)p_k = \int_0^\infty \big(S(\xi) + \xi S'(\xi)\big)\chi_k \,\mathrm{d}\xi$, we have

$$
\int_0^\infty S'(\xi) \left[\frac{\partial \chi_k}{\partial t} - \nabla\cdot(\chi_k \nabla W_k) + \chi_k \frac{W_k - \xi}{\nu} \right] \mathrm{d}\xi - \int_0^\infty \frac{S(\xi)-S(0)}{\nu} \chi_k \,\mathrm{d}\xi = \int_0^\infty S'(\xi) \mu_k(\mathrm{d}\xi). \tag{34}
$$

Since $\chi_k(\xi) = -\partial_\xi \int_\xi^\infty \chi_k(t,x;\eta)\,\mathrm{d}\eta$, integrating by parts yields

$$
\int_0^\infty \frac{S(\xi)-S(0)}{\nu} \chi_k \,\mathrm{d}\xi = \int_0^\infty \frac{S'(\xi)}{\nu} \int_\xi^\infty \chi_k(t,x;\eta)\,\mathrm{d}\eta \,\mathrm{d}\xi.
$$

Therefore, Eq. (34) is equivalent to

$$
\frac{\partial \chi_k}{\partial t} - \nabla\cdot(\chi_k \nabla W_k) + \chi_k \frac{W_k - \xi}{\nu} - \frac{1}{\nu}\int_\xi^\infty \chi_k(t,x;\eta)\,\mathrm{d}\eta = \mu_k. \tag{35}
$$

One can simplify this relation and write

$$
\frac{\partial \chi_k}{\partial t} - \nabla\cdot(\chi_k \nabla W_k) + \chi_k \frac{W_k - \xi}{\nu} - \frac{(p_k-\xi)_+}{\nu} = \mu_k.
$$

Finally, Eq. (35) is equivalent to

$$
\frac{\partial \chi_k}{\partial t} - \nabla\cdot(\chi_k \nabla W_k) + \chi_k \frac{W_k - p_k}{\nu} = \mu_k.
$$

In particular, integrating in ξ we recover the expected formula

$$
\frac{\partial p_k}{\partial t} - \nabla\cdot(p_k \nabla W_k) + \frac{p_k}{\nu}[W_k - p_k] = \int_0^\infty \mu_k \,\mathrm{d}\xi.
$$

5.3 Proof of Lemma. Step 3. Equation satisfied by f. We may pass to the limit in Eq. (35). For all $T>0$, the sequence $(\mu_k)_k$ is uniformly bounded in $L^1\big((0,T)\times\mathbb{R}^d\times\mathbb{R}\big)$ thanks to estimate from Lemma 3.5. Thus we may extract a subsequence, converging, in the weak sense of measures, towards a measure $\mu \in \mathcal{M}_b\big((0,T)\times\mathbb{R}^d\times\mathbb{R}\big)$. Due to the fact that

$$
Q_k(t,x;\xi) = W_k - \xi + \nu r_k G^{(1)}(\xi) + \nu(1-r_k)G^{(2)}(\xi),
$$

which is positive for $\xi \le p_m$, we have

$$
\mu(t,x;\xi) \ge 0,
$$

for $\xi \le p_m$. Therefore passing to the limit $k \to +\infty$ in Eq. (35), we obtain, in the sense of distributions,

$$
\frac{\partial \chi}{\partial t} - \nabla\cdot(\chi \nabla W_\infty) + \chi \frac{W_\infty - \xi}{\nu} - \frac{1}{\nu}\int_\xi^\infty \chi(t,x;\eta)\,\mathrm{d}\eta = \mu.
$$

Using Eq. (26), this equation can also be written as

$$
\frac{\partial \chi}{\partial t} - \nabla\cdot(\chi \nabla W_\infty) + \chi \frac{W_\infty - \xi}{\nu} - f(t,x)\frac{(H(r_\infty, W_\infty) - \xi)_+}{\nu} = \mu,
$$

and thus

$$
\frac{\partial \chi}{\partial t} - \nabla\cdot(\chi \nabla W_\infty) + \chi \frac{W_\infty - H(r_\infty, W_\infty)}{\nu} = \mu. \tag{36}
$$

Using the assumption of compactness of initial data, *cf.* Eq. (4b), this equation is complemented with the initial condition

$$
\chi(t=0,x;\xi) = \mathbb{1}_{\Omega^0}(x)\, \mathbb{1}_{\{0<\xi<H(r_\infty(t=0,x), W_\infty(t=0,x))\}}(\xi),
$$

and

$$
f(t=0,x) = \mathbb{1}_{\Omega^0}(x) =: f^0(x).
$$

It is useful to keep in mind the equivalent form of this equation,

$$\frac{\partial \chi}{\partial t} - \nabla \chi \cdot \nabla W_\infty + \chi \frac{p_\infty - H(r_\infty, W_\infty)}{\nu} = \mu \geq 0,$$

and thus, using Eq. (27),

$$\frac{\partial \chi}{\partial t} - \nabla \chi \cdot \nabla W_\infty + \chi\, H(r_\infty, W_\infty) \frac{f-1}{\nu} = \mu \geq 0. \tag{37}$$

We can also integrate Eq. (36) and recover

$$\frac{\partial p_\infty}{\partial t} - \nabla \cdot (p_\infty \nabla W_\infty) + \frac{p_\infty}{\nu}[W_\infty - H(r_\infty, W_\infty)] = \int_0^\infty \mu \,\mathrm{d}\xi.$$

It is useful to consider the function

$$g(t,x) = f^0\big(X_{(t,x)}(s=0)\big),$$

with the characteristics defined by

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} X_{(t,x)}(s) & = -\nabla W_\infty(s, X_{(t,x)}(s)), \\ \quad X_{(t,x)}(t) & = \mathrm{Id}\,. \end{cases}$$

This function g is a solution, in the distributional sense, of the transport equation

$$\frac{\partial g}{\partial t} - \nabla g \cdot \nabla W_\infty = 0,$$

equipped with $g^0 = f^0$. Using Eq. (37) and $0 \leq f \leq 1$, we find

$$\frac{\partial f}{\partial t} - \nabla f \cdot \nabla W_\infty = \mu(t,x;\xi) + \chi\, H(r_\infty, W_\infty) \frac{1-f}{\nu} \geq 0. \tag{38}$$

From the comparison principle, we conclude not only that $f(t,x) \geq g(t,x)$, but also that on the set $\{g(t,x) = 1\}$ there holds $f(t,x) = 1$, and, moreover, $\mu(t,x;\xi) = 0$, whenever $\xi < p_m$.

5.4 Proof of Lemma. Step 4. Identification of the function f. Another wording for the conclusion of the previous step is that

$$\Omega(t) = Y_{(x)}(t)[\Omega^0] = \{p_\infty(\cdot, t) > 0\},$$

with $Y_{(x)}(t)$ being the limiting flow of the forward flow $Y_{(x)}^{(k)}(t)$ corresponding to the velocity field $-\nabla W_k$, *i.e.*,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} Y_{(x)}^{(k)}(t) & = -\nabla W_k\Big(t, Y_{(x)}^{(k)}(t)\Big), \\ Y_{(x)}^{(k)}(t=0) & = \mathrm{Id}\,. \end{cases}$$

We point out that ∇W_k is slightly less regular than uniformly Lipschitz. Nevertheless, the flow $Y_{(x)}^{(k)}(t)$ is well-defined almost everywhere thanks to the DiPerna-Lions theory, *cf.* [15]. Observe that we have

$$p_k(t,x) = 0, \tag{39}$$

for $x \in \mathbb{R}^d \setminus \Omega^k(t)$ where, $\Omega^k(t) := Y_{(x)}^{(k)}(t)\big[\Omega^0\big]$. From Eq. (39) and the strong convergence of the flow, we infer that

$$p_\infty(t, \cdot) = 0,$$

on the set $Y_{(x)}(t)[\mathbb{R}^d \setminus \Omega^0]$. Then we have $f(t,x) = \mathbb{1}_{\Omega(t)} = \mathbb{1}_{\{p_\infty(t,x)>0\}}$. We recall that, by definition, $f = \text{weak-}^\star - \lim_{k\to+\infty} \mathbb{1}_{\{p_k \geq p_m/2\}}$.

5.5 Proof of Lemma. Step 5. Strong Convergence of the Pressure. We show that this implies the strong convergence of p_k in $L^1_{loc}((0,T)\times\mathbb{R}^d)$ towards the function $H(r_\infty, W_\infty)\mathbb{1}_{\{p_\infty>0\}}$. Let U be an open and bounded subset of $\mathbb{R}^d$, we have

$$\int_0^T\int_U \big|p_k - H(r_\infty, W_\infty)\mathbb{1}_{\{p_\infty>0\}}\big| \,\mathrm{d}x \le I_k + II_k + III_k, \tag{40}$$

with

$$I_k = \int_0^T\int_U \mathbb{1}_{\{p_k\ge p_m/2\}}|p_k - H(r_\infty, W_\infty)| \,\mathrm{d}x,$$

$$II_k = \int_0^T\int_U \mathbb{1}_{\{p_k< p_m/2\}}p_k \,\mathrm{d}x,$$

$$III_k = \int_0^T\int_U H(r_\infty, W_\infty)\big(\mathbb{1}_{\{p_k\ge p_m/2\}}(1-\mathbb{1}_{\{p_\infty>0\}}) + \mathbb{1}_{\{p_k< p_m/2\}}\mathbb{1}_{\{p_\infty>0\}}\big) \,\mathrm{d}x.$$

We have already shown that I_k and II_k converge to zero as $k\to\infty$, *cf.* Eq. (32) and Eq. (30), respectively.
Using the fact that W_∞ is bounded in L^∞, the last term may be estimated

$$III_k \le C\int_U \Big(\mathbb{1}_{\{p_k\ge p_m/2\}}(1-\mathbb{1}_{\{p_\infty>0\}}) + (1-\mathbb{1}_{\{p_k\ge p_m/2\}})\mathbb{1}_{\{p_\infty>0\}}\Big) \,\mathrm{d}x,$$

for some non-negative constant C. We have shown above that $\mathbb{1}_{\{p_k\ge p_m/2\}}$ converges weakly towards $\mathbb{1}_{\{p_\infty>0\}}$. Then passing to the limit $k\to+\infty$ in the latter inequality, we deduce that $\lim_{k\to+\infty} III_k = 0$. We conclude from the inequality in Eq. (40) that

$$\int_0^T\int_U |p_k - H(r_\infty, W_\infty)\mathbb{1}_{\{p_\infty>0\}}| \,\mathrm{d}x \underset{k\to+\infty}{\longrightarrow} 0,$$

for any open and bounded set $U\subset\mathbb{R}^d$. By uniqueness of the weak limit, we deduce that $p_\infty = H(r_\infty, W_\infty)\mathbb{1}_{\{p_\infty>0\}}$ almost everywhere in $(0,T)\times\mathbb{R}^d$.

Finally, from definition of the measures μ_k, *cf.* Eq. (33), the limit in Eq. (24) is now a consequence of

$$k\int_0^T\int_{\mathbb{R}^d} p_k(t,x)|Q_k(t,x)| \,\mathrm{d}x\,\mathrm{d}t = \frac{k}{k-1}\int_0^T\int_{\mathbb{R}^d}\int_0^\infty |\mu_k(t,x;\xi)| \,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,$$

and the fact that μ_k vanishes for $k\to\infty$. Indeed, recall that from Eq. (38) we infer that $\mu = 0$ both when $f = 1$ and $f = 0$. Therefore, we recover Eq. (24) which concludes the proof of Lemma 5.1.

6 Incompressible Limit and Complementarity Relation

Now, the proof of the Theorem 2.3 is easily deduced from Lemma 5.1. First, up to a subsequence, we have that p_k converges almost everywhere towards p_∞. On the one hand, recalling that the sequence $(p_k)_k$ is uniformly bounded in L^∞, we use the Lebesgue dominated convergence Theorem to show that, for any open and bounded set U,

$$\int_0^T\int_U p_k\Big|W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1-r_k)G^{(2)}(p_k)\Big| \,\mathrm{d}x\,\mathrm{d}t$$
$$\underset{k\to+\infty}{\longrightarrow} \int_0^T\int_U p_\infty\Big|W_\infty - p_\infty + \nu r_\infty G^{(1)}(p_\infty) + \nu(1-r_\infty)G^{(2)}(p_\infty)\Big| \,\mathrm{d}x\,\mathrm{d}t.$$

On the other hand, from estimate Lemma 3.5 we have that

$$\int_0^T\int_U p_k\Big|W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1-r_k)G^{(2)}(p_k)\Big| \,\mathrm{d}x\,\mathrm{d}t \underset{k\to+\infty}{\longrightarrow} 0.$$

We deduce that

$$p_\infty\Big(p_\infty - W_\infty - \nu n_\infty^{(1)}G^{(1)}(p_\infty) - \nu n_\infty^{(2)}G^{(2)}(p_\infty)\Big) = 0,$$

almost everywhere in $(0,T)\times\mathbb{R}^d$, which is the required complementarity relation.
Since the sequences $(n_k^{(i)})_k$, $(G(p_k))_k$ and $(W_k)_k$ converge strongly, we readily pass to the limit in Eq. (3a) and Eq. (3b), to obtain the claimed limiting equations. Finally, passing to the limit in the relation,

$$n_k p_k = \left(\frac{k-1}{k}\right)^{1/(k-1)} p_k^{k/(k-1)},$$

we deduce that $(1-n_\infty)p_\infty = 0$.

Keeping Lemma 3.3 in mind, we know that phase segregation is preserved by the equation, *i.e.*, initially segregated data remains segregated for all times. This allows us to characterise the free boundary motion more precisely.

Remark 6.1 (Segregation Patches)**.** In the case of segregated densities the limiting equation may be simplified on each populated patch. Indeed, we can define three disjoint subsets

$$\Omega_0 = \{p_\infty = 0\}, \quad \Omega_1 = \{p_\infty > 0,\, n_\infty^{(1)} = 1\}, \quad \text{and} \quad \Omega_2 = \{p_\infty > 0,\, n_\infty^{(2)} = 1\}.$$

On Ω_0, the limiting system reduces to $-\nu\Delta W_\infty + W_\infty = 0$, coupled with the equations for $n_\infty^{(i)}$. This, in turn, implies that $\nabla W_\infty = 0$, whence

$$\frac{\partial n_\infty^{(i)}}{\partial t} = n_\infty^{(i)} G^{(i)}(0),$$

i.e., the two species grow exponentially, yet at different rates, depending on the growth term, in the absence of pressure.

On the sets Ω_i, for $i=1,2$, the complementarity relation takes on the form $W_\infty + \nu G^{(i)}(p_\infty) = 0$, *i.e.*, $p_\infty = \left(G^{(i)}\right)^{-1}(-W_\infty/\nu)$, and it remains to solve the elliptic equation

$$-\nu\Delta W_\infty + W_\infty = (G^{(i)})^{-1}(-W_\infty/\nu),$$

on each patch Ω_i, $i=1,2$.

In summary, we only have to solve the elliptic equation

$$-\nu\Delta W_\infty + F(W_\infty) = 0,$$

where

$$F(W_\infty) = \begin{cases} W_\infty, & \text{on } \Omega_0, \\ W_\infty - (G^{(i)})^{-1}(-W_\infty/\nu), & \text{on } \Omega_i,\ i=1,2. \end{cases}$$

The speed of each interface is given by $v = -\nabla W_\infty$.

7 Conclusions

In this work we establish the incompressible limit for a viscous two-species tissue model in arbitrary dimension. The two species are linked through growth and death processes and an elliptic equation, the so-called Brinkman law. In addition, we were able to show the phase segregation property, *i.e.*, the fact that the two species remain segregated if they were segregated initially. Our results extend those in the literature to higher dimensions and to two species. Albeit technical, the proof hinges on three key observations: the transport of regularity à la Bresch & Jabin, the observation of incorporating the total population as in [14] as an additional or "auxiliary" variable, and the kinetic formulation of [25]. We stress that no additional information is gained from the estimates on the total population, on the contrary, its purpose is to make up for a change in regularity in the individual species.
The same strategy may also be employed in order to perform the rigorous limit for cross-reacting species, for instance, to account for a fraction of normal cells that become abnormal upon cell division. As a matter of fact, the a priori estimates are obtained in the same way, and estimating the two individual species and the total population simultaneously leads to the same beneficial cancellations. Finally, the kinetic reformulation requires a re-definition of the

function H, however its core properties remain unchanged. Therefore no additional conceptual difficulties are to be encountered.
While the result of this paper answers an important question it gives rise to a range of new and interesting problems. One cannot help but wonder as to whether the compactness of the initial data is, indeed, necessary. In the one species case, this is not a requirement, and the equation has, in some sense, an intrinsic regularising effect. Since the method of BRESCH & JABIN has a flavour of propagating compactness the initial compactness is a strong requirement. It would therefore be interesting to understand if this requirement is only necessary for this type of argument or if it can be removed.

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APPENDIX

For the readers' convenience we provide a short and formal recapitulation of the compactness criterion employed in this work to propagated the compactness of the two individual species as well as the total population.
To the best of the authors' knowledge, the method was first introduced in [2] in the context of nonlinear transport equations. It hinges on the observation that (local) compactness in Lebesgue spaces is equivalent to controlling quantities like

$$\frac{1}{|\log h|}\int_{\mathbb{R}^d} K_h(x-y)|n(x)-n(y)|^p \,\mathrm{d}x\,\mathrm{d}y, \tag{41}$$

as $h \to 0$, for certain families of kernels K_h. In particular, compactness in the sense of the well-known Frechet-Kolmogorov-Riesz shift property is implied as it can be shown that

$$\int_{\mathbb{R}^d} |n(x-h)-n(x)|^p \,\mathrm{d}x \sim \frac{1}{|\log h|}\int_{\mathbb{R}^d} K_h(x-y)|n(x)-n(y)|^p \,\mathrm{d}x\,\mathrm{d}y + C|h|. \tag{42}$$

For more details and a rigorous argument we refer the reader to [2, Lemma 3.1]. The key observation of their paper is that this quantity is propagated along the flow induced by the transport equations under consideration. A subsequent Gronwall-type estimate allows them to infer the compactness of solutions at any time solely from the compactness of the initial data.
With this formal motivation at hand, we shall now present a possible construction of a suitable family of kernels as in the statement of Lemma 4.4, while refering the reader to [8, Section 4] for further guidance.
As in the main body, we denote by $B_{\epsilon_0}(x_0)$ the Euclidean ball centred at the point $x_0 \in \mathbb{R}^d$ with radius $\epsilon_0 > 0$. Next, we define an auxiliary family, $(K_h)_{h>0}$, of smooth, non-negative, symmetric functions given by

$$K_h(x) = \frac{c_h}{(|x|^2+h^2)^{d/2}}.$$

The constants c_h are chosen in such a way that each K_h is normalised, *i.e.*, $\|K_h\|_{L^1(\mathbb{R}^d)} = 1$. One can then obtain the inequality

$$|x||\nabla K_h(x)| \le C K_h(x), \tag{43}$$

which holds for some non-negative constant C, independent of h, due to the specific choice of K_h. We remark that this property is not essential as far as the compactness criterion is concerned. However, applied to conservation laws, the property is indispensable for this allows to estimate the divergence of the flux after an integration by parts. Finally, we introduce the family of functions

$$\mathcal{K}_h(x) = \int_h^1 K_s(x) \frac{\mathrm{d}s}{s},$$

which has the following properties

$$\|\mathcal{K}_h\|_{L^1(\mathbb{R}^d)} \sim |\log h|,$$

and, for each $\eta > 0$,

$$\sup_h \int_{\{|x| \geq \eta\}} \mathcal{K}_h \,\mathrm{d}x < \infty.$$

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